

Past and future blurring at fundamental length scale

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We obtain the κ -deformed versions of the retarded and advanced Green functions and show that their causality properties are blurred in a time interval of the order of a length parameter $q = 1/(2\kappa)$. The functions also indicate a smearing of the light cone. These results favor the interpretation of q as a fundamental length scale below which the concept of a point in spacetime should be substituted by the concept of a fuzzy region of radius q , as proposed long ago by Heisenberg.

In the early days of quantum field theory (QFT) Heisenberg analyzed several arguments for the introduction of a universal invariant length parameter in physics [1], say q , which should be added to the fundamental speed c and the fundamental action \hbar . One of Heisenberg's arguments relies upon March's [2] proposal that geometry should be modified at small lengths. Heisenberg argues that this length establishes an upper limit in the accuracy of measuring positions and that the concept of length itself can be used without restriction only for distances which are large compared to the fundamental length. The invariant character of such fundamental length follows then from the relativistic principle [3]. It is well known that the combination of gravitational and quantum effects implies indeed the existence of such a length [4], usually at Planck scale, *i.e.*, of order 10^{-33} cm, and several recent proposals have supported the idea of introducing such a fundamental length (or the corresponding fundamental energy) (see, *e.g.* [3, 5] and the reviews [6]). A fundamental length of the order of 10^{-32} cm has also been advocated as a consequence of string theory (for a review, see [7]), in a way which suggests that below such a length "the smaller distances are not there", to use Witten's words [7]. In this context it is natural to substitute the concept of a point in space by a fuzzy region of radius q and the concept of an instant of time by a fuzzy time interval of duration q/c . More generally, a point in spacetime would be substituted by a fuzzy region of radius q , if we used natural units, as we do from now on. A quite remarkable implication of such a view is a relaxation of the concept of causality, since a portion of immediate future or past time of the order of q , for example, would be indistinguishable from the present time. To face this rather far from intuitive concept we may cite Heisenberg's remark that the fundamental length together with the constants c and \hbar designate "the limits in whose proximity our intuitive concepts can no longer be used without misgivings" [1]. A way of introducing a fundamental length q at which our intuitive concept of spacetime point becomes fuzzy is to implement a non-commutative geometry of spacetime [8] by promoting the spacetime coordinates to hermitian operators obeying relations of the form $[x^\mu, x^\nu] = iq^2\theta^{\mu\nu}$, in which we take $\theta^{\mu\nu}$ dimensionless to make explicit the presence of q^2 in the commutation relations. In the original formulation of Snyder [9] $q^2\theta^{\mu\nu}$ is postulated to be

an operator of the Lorentz algebra, but more recently the non-commutativity of space coordinates has been obtained as an appropriate limit of string theory in non-trivial backgrounds [10]. These non-commutative theories present nonlocality in spatial directions which ruins Lorentz invariance and in the case of non-commutativity of space coordinates and time acausal effects may also appear [11, 12]. Actually, it can be argued on general grounds that violation of Lorentz invariance or locality lead to violations of unitarity or causality [13]. Another way of introducing a fundamental length q is to deform the usual Lorentz invariant dispersion relation in usual spacetime [6]. We will consider here such a deformation to investigate its effect on causality properties. It is certainly worthwhile to investigate if a theory containing a fundamental length q shows plausible and expected properties or unavoidable inconsistencies. For instance, we could check if such a theory exhibits the above mentioned "relaxed causality" due to the existence of a fundamental length q . The appropriate quantities to investigate these matters of causality are the Green functions for signal propagation, namely, in the usual case, the retarded and advanced Green functions of the wave operator $(\partial/\partial t)^2 - \nabla^2$. In this letter, we consider a theory in which such operator is substituted by the deformed operator depending on a length parameter q , given by $q^{-2}\sin^2(q\partial/\partial t) - \nabla^2$, whose origin will be explained below. Note that this operator reduces to the usual wave operator in the limit $q \rightarrow 0$. A Green function associated to this deformed operator satisfies the differential equation

$$\left[\frac{1}{q^2} \sin^2\left(q \frac{\partial}{\partial t}\right) - \nabla^2 \right] G(\mathbf{x}, t; \mathbf{x}', t') = -\delta^3(\mathbf{x} - \mathbf{x}') \delta(t - t'). \quad (1)$$

At first sight there is no reason for this Green function to exhibit the above mentioned property of blurring the concepts of past and future inside time intervals of order q . In other words, there is no *a priori* reason for the parameter q in (1) to play the role of a fundamental length. The motivation to investigate this possibility is given by the origin of this particular deformed operator, to which we pass now.

Due to the assumed smallness of the fundamental length q , a theory describing phenomena at scale q should be at least a theory of quantum fields, if not of more fundamental objects. A natural place to introduce a fun-

damental length in QFT is in its algebra of space-time symmetries, since the fundamental length q is obviously a property of space-time. In usual relativistic QFT the algebra of space-time symmetries is the Poincaré Lie algebra, defined by the commutation relations of the generators P^0 and \mathbf{P} of time and space translations, as well as the generators of space rotations and boosts. A fundamental property of this algebra is that $\mathbf{P}^2 - P_0^2$ is a Casimir invariant, *i.e.*, it commutes with all the generators of the algebra. From this invariant we obtain the wave operator by the usual substitutions $P^0 \mapsto i\partial/\partial t$ and $\mathbf{P} \mapsto -i\nabla$. The introduction of a length q in this algebra can be done by a deformation of the algebra. The mathematical procedure of deforming an algebra consists in defining a new algebra depending on the parameter q with the property that the original one is recovered in the limit $q \rightarrow 0$ of no deformation. In this context q is called the deformation parameter. The deformation of the Poincaré Lie algebra can be done in such a way that the deformed algebra is part of a mathematical structure called Hopf algebra, also known as quantum group or quantum algebra (see, *e.g.* [14, 15]). This is a mathematical object used to describe generalized symmetries not properly described by Lie algebras [14]. A deformation of this kind has been obtained by Lukierski, Ruegg, Nowicki and Tolstoy [16, 17] and is called κ -deformed Poincaré algebra, from the κ used to denote the deformation parameter with dimension of mass. Among the remarkable properties of the κ -deformation of the Poincaré algebra one is especially important here, namely: it is the simplest deformation of the Poincaré algebra which gives rise to a quantum group which provides a theoretical laboratory for QFT investigations [18]. The κ -deformation of the Poincaré algebra is defined by commutation relations between generators P^0 and \mathbf{P} of time and space translations, as well as the generators of space rotations and boosts [16, 17] and has the Casimir invariant $\mathbf{P}^2 - (2\kappa)^2 \sinh^2[P_0/(2\kappa)]$. Using in this expression the usual substitutions $P^0 \mapsto i\partial/\partial t$ and $\mathbf{P} \mapsto -i\nabla$ we obtain the κ -deformed wave operator $(2\kappa)^2 \sin^2[(2\kappa)^{-1}\partial/\partial t] - \nabla^2$. The trivial substitution

in this operator of the parameter κ by the length parameter $q = (2\kappa)^{-1}$ leads to the deformed operator in (1). Regardless of these motivations for adopting here this deformed operator, we should observe that all the calculations and investigation of causality questions of this letter follow from the sole assumption of equation (1). Let us, then, calculate the κ -deformed retarded and advanced Green functions. We start by considering the following Fourier transform of the Green function in (1):

$$G^{(\gamma)}(\mathbf{x}, t; \mathbf{x}', t') = \int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} \mathcal{G}^{(\gamma)}(\mathbf{k}, t-t'), \quad (2)$$

where

$$\mathcal{G}^{(\gamma)}(\mathbf{k}, t-t') = \int_{\gamma} \frac{d\omega}{2\pi} \frac{e^{-i\omega(t-t')}}{q^{-2} \sinh^2(q\omega) - \mathbf{k}^2}, \quad (3)$$

and γ is a contour in the complex ω -plane infinitesimally close to the real axis but bypassing the singularities in this axis. The poles on the complex ω -plane are given by $\omega_n = \pm\omega_{\mathbf{k}} + 2\pi in/q$ ($n \in \mathbb{Z}$), where $\omega_{\mathbf{k}} = q^{-1} \sinh^{-1}(q|\mathbf{k}|)$. The choice of γ bypassing the poles at points $\omega = \pm\omega_{\mathbf{k}}$ on the real axis is equivalent to displace them by infinitesimal imaginary quantities. The displacement $\pm\omega_{\mathbf{k}} \mapsto \pm\omega_{\mathbf{k}} - i\varepsilon$ gives rise to the retarded Green function of the wave equation and the displacement $\pm\omega_{\mathbf{k}} \mapsto \pm\omega_{\mathbf{k}} + i\varepsilon$, to the advanced Green function of the wave equation. These Green functions can be calculated by the usual complex variable technique of closing the contour γ by an infinite semicircle on the upper or lower half plane. For the κ -deformed retarded prescription we take the poles at $\omega_n = \pm\omega_{\mathbf{k}} + 2\pi in/q - i\varepsilon$ ($n \in \mathbb{Z}$). For convenience, the poles originally outside the real axis have also been displaced by $-i\varepsilon$, which is of no consequence due to the usual limit $\varepsilon \rightarrow 0$ taken at the end of the calculations. We replace the symbol (γ) by $(+)$ or $(-)$ in (2) and (3) to indicate that the κ -deformed retarded or advanced prescription has been enforced on the Green function. Now, by writing the hyperbolic sine in (3) in terms of exponentials and changing the integration variable to $z = 2q\omega$, we obtain in the κ -retarded case:

$$\mathcal{G}^{(+)}(\mathbf{k}, t-t') = \frac{q}{\sinh(2q\omega_{\mathbf{k}})} \int_{-\infty}^{\infty} \frac{dz}{2\pi} e^{-i\frac{(t-t')}{2q}z} \frac{d}{dz} \log \left[e^z - e^{-2q(\omega_{\mathbf{k}}+i\varepsilon)} \right] - \left\{ i\varepsilon \longrightarrow -i\varepsilon \right\}. \quad (4)$$

These integrals are in a form ready for an application of Cauchy's integral [19], also known as the Argument Principle. For $t-t' > 0$ the integrations on the real axis in the two integrals (4) are extended to a closed contour with the usual infinite-radius semicircle on the lower imagi-

nary half-plane, while for $t-t' < 0$, the contour is closed by an infinite-radius semicircle on the upper imaginary half-plane. A straightforward calculation gives us for the Green function (4) in the reciprocal space the expression

$$\mathcal{G}^{(+)}(\mathbf{k}, t-t') = -q \frac{\sin[\omega_{\mathbf{k}}(t-t')]}{\sinh(2q\omega_{\mathbf{k}})} \left\{ \frac{\Theta(t-t')}{1 - e^{-\frac{2\pi(t-t')}{q}}} - \Theta(t'-t) \frac{1}{2} \left[\coth\left(\frac{\pi(t'-t)}{q}\right) - 1 \right] \right\}. \quad (5)$$

Substituting (5) in (2) and performing the angular inte-

gral, we obtain

$$G^{(+)}(x, x') = \frac{-1}{4\pi} \left\{ \frac{\Theta(t-t')}{1 - e^{-\frac{2\pi(t-t')}{q}}} + \Theta(t'-t) \frac{1}{2} \left[\coth\left(\frac{\pi(t'-t)}{q}\right) - 1 \right] \right\} \\ \times \frac{1}{|\mathbf{x} - \mathbf{x}'|} \int_0^\infty \frac{2 dk}{\pi \sqrt{1 + q^2 k^2}} \sin \left[\frac{|t-t'|}{q} \sinh^{-1}(qk) \right] \sin(k|\mathbf{x} - \mathbf{x}'|) , \quad (6)$$

which is the κ -retarded Green function in space-time in terms of a single quadrature. In the limit $q \rightarrow 0$ of no deformation the term containing $\Theta(t'-t)$ goes to zero, the coefficient of $\Theta(t-t')$ reduces to 1 and the integral tends to a Dirac delta function. In this way we obtain from (6) the correct non-deformed limit $G^{(+)}(x, x')|_{q=0} = \frac{-\Theta(t-t')}{4\pi|\mathbf{x}-\mathbf{x}'|} \delta[t' - (t - |\mathbf{x} - \mathbf{x}'|)]$. The κ -retarded Green function (6) also exhibits the zero limit for the remote past, namely, $G^{(+)}(x, x') = 0$ for $t \rightarrow -\infty$, as in the non-deformed case. However, in contrast with the non-deformed case, the κ -retarded Green function (6) has also contribution from the future, stemming from the term containing $\Theta(t'-t)$ in (6). It is interesting to notice that the existence of such a contribution from the future is natural if we consider the deformed dispersion relation which leads to (1) as describing phenomena at a fundamental space-time scale q , at which the concepts of point and instant lose their significance and should be substituted by the concept of diffuse regions of space-time with dimensions of the order of q . At this scale it is nothing but natural that distinction between past and future loses its meaning in a time interval of the order of

q , thereby giving the reason for the presence of the term containing $\Theta(t'-t)$ in the κ -retarded Green function (6). However, this interpretation must pass the consistency checking that this contribution from the future must die out rapidly when the measurement time is much greater than the fundamental length q , *i.e.*, when $t' - t \gg q$. This is by far satisfied by the κ -retarded Green function (6), due to the property

$$G^{(+)}(\mathbf{x}, t; \mathbf{x}', t') \Big|_{t'-t \gg q} \sim e^{-2\pi(t'-t)/q} \sim 0 . \quad (7)$$

Expression (6) for the κ -retarded Green function with these properties, together with the analogous results for the κ -advanced Green function, are the main result of this letter. We have seen that the κ -retarded Green function favors the idea that at a fundamental time scale q the concept of instant of time should be substituted by the concept of a fuzzy time interval of the order of q . It is also interesting to ask if the κ -deformed Green functions describe some sort of fuzziness in space directions. In the Green functions this information appears as the effect of the deformation upon the usual light cone described by the Dirac delta function. To address this question it is convenient to write the κ -retarded Green function (6) as

$$G^{(+)}(x, x') = \frac{-\Theta_q(t-t')}{4\pi|\mathbf{x} - \mathbf{x}'|} \delta_q(x - x') ; \quad \delta_q(x - x') = \int_0^\infty \frac{2 dk}{\pi \sqrt{1 + q^2 k^2}} \sin \left[\frac{|t-t'|}{q} \sinh^{-1}(qk) \right] \sin(k|\mathbf{x} - \mathbf{x}'|) , \quad (8)$$

where we have defined $\Theta_q(t-t')$ as the combination of Heaviside functions between braces in (6) and $\delta_q(x - x')$ as the integral in (6). Since $\lim_{q \rightarrow 0} \delta_q(x - x') = \delta(|t-t'| - |\mathbf{x} - \mathbf{x}'|)$, the integral $\delta_q(x - x')$ defined in (8) itself is a smeared Dirac delta function representing a fuzzy light cone in the deformed case. Naturally, the smaller is q compared to $|t-t'|$ and $|\mathbf{x} - \mathbf{x}'|$, the sharper is the deformed light cone. The fuzzy light cone is clearly illustrated in Fig. 1 as the plot of the integral $\delta_q(x - x')$ in (8) (for $\mathbf{x} = \mathbf{0}$, $t = 0$, $x'^2 = x'^3 = 0$ and $x'^1 = x'$). Since $q = 0.001$ AU in this figure, the smeared light cone shows a definitely regular appearance for x and t greater than 0.5 AU. In Fig. 2 a contour plot depicts a light cone leg in a range of $6000q$ with constant width and height except for small oscillations to be expected from the nature of the integrand in (8).

For the κ -retarded Green function the smeared future light cone is exponentially suppressed according to our fundamental result (7), while the smeared past light cone is multiplied by the coefficient of $\Theta(t-t')$ in (6), which tends to 1 for $t - t' \gg q$. The little bumps outside the smeared light cone and this smearing itself describe the Lorentz violation induced by the deformation, which are negligible in the regime in which q is considered very small. Summing up, the κ -deformation smears the light cone in spacetime and provides for the κ -retarded Green function contributions from the past and an expected contribution from an interval of future of the order of q . It is interesting to compare the smearing of the light cone due to κ -deformation with the change from the light cone to the so called light wedge occurring in non commutative space [12, 20].

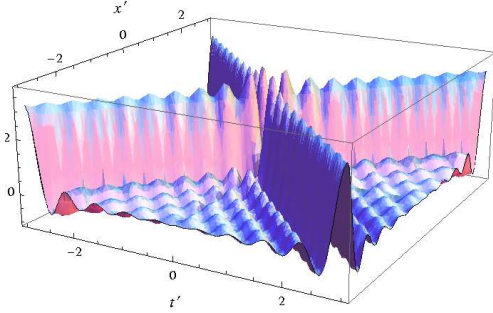


FIG. 1. A plot of δ_q given in (8) with $q = 0.001$ AU.

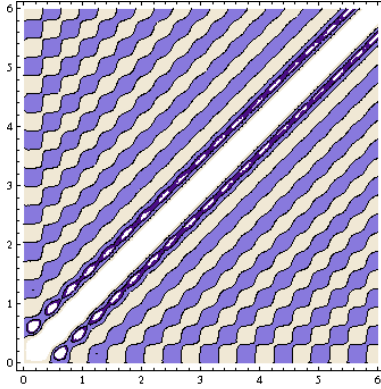


FIG. 2. A contour plot with contours at -0.5, 0 and 3.0 AU (darker at lower levels) of a light cone leg.

A completely analogous calculation leads, for the κ -advanced Green function, to the result $G^{(-)}(x, x') = G^{(+)}(x', x)$ with the correct non-deformed limit, $G^{(-)}(x, x')|_{q=0} = G^{(+)}(x', x)|_{q=0}$, and satisfying, in

the distant future, $\lim_{t \rightarrow \infty} G^{(-)}(x, x') = 0$. In the κ -advanced Green function the contributions from the blurred light cone, as expected, come from both future and past. The contribution from the past comes from an interval of time of the order of q with an exponential decay for measurement times much greater than this interval, $G^{(-)}(x, x')|_{t-t' \gg q} \sim e^{-2\pi(t-t')/q} \sim 0$. Naturally, we can make comments on this property in analogy with the ones made for the property (7) in the κ -retarded case.

In this letter we presented the Green functions with the retarded and advanced prescription for the κ -deformed dispersion relation. These κ -retarded and κ -advanced Green functions have the desired limits when the deformation goes to zero. Moreover, in the case of small deformation, the deformed Green functions exhibit a blurring in the distinction between past and future inside time intervals of the order of the fundamental length parameter q . In this way the κ deformation appears as an appropriate tool to describe the possibility of past and future blurring at a fundamental length scale. This is a highly desirable feature if we want to consider the deformation and its length parameter q as describing new physics at extremely small scale, say at the Planck scale. We have also seen that the κ -deformation also produces a blurring in the light cone. It is rather surprising that objects such as those κ -deformed Green functions, which can be defined from the sole data of a deformed dispersion relation, exhibit the relaxed causality behavior to be expected in the realm of QFT. It would be interesting to check if other deformed theories, as for instance, the theory based on the κ -Poincaré algebra in the bicrossproduct basis [18, 21], also exhibited the nice “relaxed causality” properties described here.

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